

PHYSICS 513, QUANTUM FIELD THEORY
Homework 1

Due Tuesday, 9th September 2003

JACOB LEWIS BOURJAILY

Problem 1) The conservation of four-momentum implies that in particle one's rest frame,

$$p_1^0 = m_1 = E_2 + E_3. \quad (1.1)$$

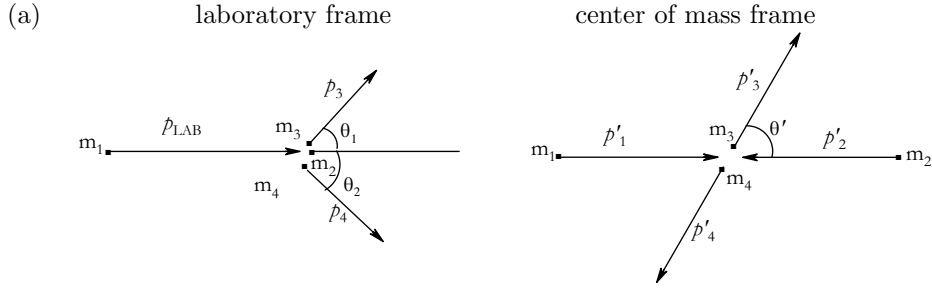
By the invariance of p_1^2 , p_2^2 , and p_3^2 , it is clear that,

$$\begin{aligned} p_1^2 = m_1^2 &= (p_2 + p_3)^2, \\ &= p_2^2 + p_3^2 + 2p_2 p_3, \\ &= m_2^2 + m_3^2 + 2E_2 E_3 - \vec{p}_2 \vec{p}_3. \end{aligned}$$

But in particle one's rest frame, $\vec{p}_2 = -\vec{p}_3$ and by (1.1), $E_3 = m_1 - E_2$. Therefore,

$$\begin{aligned} m_1^2 &= m_2^2 + m_3^2 + 2m_1 E_2 - 2(E_2^2 - \vec{p}_2^2), \\ &= m_3^2 - m_2^2 + 2m_1 E_2, \\ \therefore E_2 &= \frac{m_1}{2} + \frac{m_2^2 - m_3^2}{2m_1}. \end{aligned} \quad (1.2)$$

Problem 2)



(b) In the center of mass frame of reference, the total 4-momentum can be described by,

$$p_{cm} = p'_1 + p'_2 = (E_1 + E_2; \vec{0}) \equiv (E_{cm}; \vec{0}).$$

Note that $p_1 p_2$ is an invariant scalar product. Evaluated in the laboratory frame,

$$p_1 p_2 = E_L m_2 - \vec{p}_L \vec{0} = E_L m_2.$$

This allows us to conclude that,

$$\begin{aligned} p_{cm}^2 = E_{cm}^2 &= p_1'^2 + p_2'^2 + 2p_1 p_2, \\ \therefore E_{cm}^2 &= m_1^2 + m_2^2 + 2E_L m_2. \end{aligned} \quad (2.1)$$

(c) Consider the four-vectors η and λ defined by,

$$\begin{aligned} \eta &\equiv (p_1 + p_2) = (E_L + m_2; \vec{p}_L) & \eta' &\equiv (E'_1 + E'_2; \vec{0}) = (E_{cm}; \vec{0}); \\ \lambda &\equiv (p_1 - p_2) = (E_L - m_2; \vec{p}_L) & \lambda' &\equiv (E'_1 - E'_2; 2\vec{p}'). \end{aligned}$$

By the frame-invariance of the scalar product,

$$\eta \lambda = \eta' \lambda' = E_L^2 - m_2^2 - |\vec{p}_L|^2 = E_{cm}(E'_1 - E'_2). \quad (2.2)$$

Now consider the identity $\eta'^2\lambda'^2 = \eta^2\lambda^2$. Calculating these products and using the result above,

$$\begin{aligned}\eta'^2\lambda'^2 &= E_{cm}^2 ((E'_1 - E'_2)^2 - 4|\vec{p}'_1|^2) = ((E_L + m_2)^2 - |\vec{p}_L|^2) ((E_L - m_2)^2 - |\vec{p}_L|^2) = \eta^2\lambda^2, \\ E_{cm}^2 (E'_1 - E'_2)^2 - 4|\vec{p}'_1|^2 E_{cm}^2 &= (E_L^2 - m_2^2 - |\vec{p}_L|^2)^2 - 4m_2^2 |\vec{p}_L|^2, \\ &= E_{cm}^2 (E'_1 - E'_2)^2 - 4m_2^2 |\vec{p}_L|^2, \\ \therefore |\vec{p}'_1|^2 &= \frac{m_2^2 |\vec{p}_L|^2}{E_{cm}^2} \Rightarrow |\vec{p}'_1| = \frac{m_2 |\vec{p}_L|}{E_{cm}}.\end{aligned}\tag{2.3}$$

(d) By the conservation of four-momentum, $q = p_1 - p_3 = p_4 - p_2$. So,

$$\begin{aligned}q^2 &= (p_4 - p_2)^2 = 2m_4^2 - 2p_2 p_4, \\ &= 2m_4^2 - 2E_4 m_4, \\ \therefore q^2 &= -2m_4(E_4 - m_4).\end{aligned}\tag{2.4}$$

(e) The first part of this problem, namely that $s \equiv (p_1 + p_2)^2 = E_{cm}^2$, was demonstrated and used in part (b) above. Let us now consider $t \equiv q^2$,

$$t \equiv q^2 = p_1^2 + p_3^2 - 2p_1 p_3 = 2m_1^2 - 2E'_1 E'_3 + 2|\vec{p}'_1| |\vec{p}'_3| \cos(\theta').$$

Here, we wrote $p_1 p_2$ explicitly in the center of mass frame. Because it is an invariant, any frame will do. Now we can use the fact that $m_1 = m_3$ and $m_2 = m_4$ to see that $|\vec{p}'_1| = |\vec{p}'_3|$ and that $E'_1 = E'_3$ by using part (c) from above. We will now use the notation of the assignment where $|\vec{p}'_1| = p'$. This quickly reduces the above equality to

$$q^2 = 2(m_1^2 - E_1'^2 + p'^2 \cos(\theta')).$$

This can be simplified in two ways. First, notice that $m_1^2 - E_1'^2 = -p'^2$ because $E_1'^2 - p'^2 = m_1^2$. Second we will use the trigonometric identity $1 - \cos(\alpha) = 2\sin^2(\alpha/2)$. Introducing these simplifications we obtain

$$q^2 = -4p'^2 \sin^2(\theta'/2).\tag{2.5}$$

(f) To explore new areas of physics at very high energies, one requires the greatest center of mass energy possible. This is because the center of mass energy is what is available to create new matter in a collision. It is simple to show that fixed-target experiments have significantly lower energy than comparable colliders. This is seen by solving the expression for s in part (e) above. In a fixed target collision, we can compute $(p_1 + p_2)^2$ in the laboratory frame because it is an invariant. In the laboratory frame, $p_1 = (E_B; \vec{p}_L)$ and $p_2 = (m_2, \vec{0})$. Therefore in a fixed target experiment,

$$E_{cm}^2 = p_1^2 + p_2^2 + 2p_1 p_2 = m_1^2 + m_2^2 + 2m_2 E_B.\tag{2.6}$$

Approximating this in the case of a high energy collision where $E_B \gg m_1, m_2$,

$$E_{cm} \simeq \sqrt{2m_2 E_B}.\tag{2.7}$$

This does not look very cost effective. If you increased the beam energy 100 times, there would only be 10 times more energy available for particle creation. In the center of mass collision, however, we see that there is much higher efficiency. In such a collision, $p_1 = (E_B; \vec{p})$ and $p_2 = (E_B; -\vec{p})$. Taking the same approximation that the beam energy is significantly higher than the rest-masses of the particles involved,

$$E_{cm} \simeq 2E_B.\tag{2.8}$$

It is clear that this would be the preferred experiment. A 100 fold increase in beam energy would result in 100 times more energy available: the way one would expect it to be. Despite the energy efficiency of center of mass colliders, many experiments still use fixed target experiments. Why? There are several primary reasons. The first is that it is extraordinarily difficult and usually very expensive to build a collider. If the collider is to work with matter and antimatter like Fermilab today, LEP I or LEP II, one can use the same (vertical) magnetic fields to accelerate the particles and antiparticles in opposite directions. This saves money on magnets but requires solving enormous engineering obstacles. In the LEP accelerator at CERN, for example, both the e^- and e^+ beams were in the same vacuum chamber; they had to be prevented from interacting except in very explicit locations

along the accelerator. Imagine ultra-relativistic beams of positrons and electrons moving oppositely in a small vacuum tube only separated by a centimeter. It clearly takes a great deal of forethought.

In addition to engineering hurdles, there are also very large costs involved in building these accelerators. If the collider is built to accelerate only matter, then the same magnetic field cannot be used to accelerate opposing beams. This means that literally two entire magnetic tracks must be built (essentially two entirely separate accelerators). This is what is being done for the Large Hadron Collider at CERN.

Problem 3) We would like to consider the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - a\phi - \frac{b}{2}\phi^2 - \frac{\alpha}{3!}\phi^3 - \frac{\beta}{4!}\phi^4,$$

under the transformation $\phi \rightarrow \phi' = \phi + c$. By direct calculation,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - c \left(a + \frac{bc}{2} + \frac{\alpha c^2}{6} + \frac{\beta c^3}{24} \right) \\ &\quad - \phi \left(a + bc + \frac{\alpha c^2}{2} + \frac{\beta c^3}{6} \right) \\ &\quad - \phi^2 \left(\frac{b}{2} + \frac{\alpha c}{2} + \frac{\beta c^2}{4} \right) \\ &\quad - \phi^3 \left(\frac{\alpha}{6} + \frac{\beta c}{6} \right) \\ &\quad - \phi^4 \frac{\beta}{4!}. \end{aligned}$$

We are to show that a constant c can be chosen to remove the linear term in the Lagrangian. Notice that the constant term in the Lagrangian is fine—we can always shift the Lagrangian density by a constant without changing the equations of motion. Therefore, we must show that we can find a c such that,

$$\left(a + bc + \frac{\alpha c^2}{2} + \frac{\beta c^3}{6} \right) = 0; \quad (3.1)$$

Although it would be a terrible headache to solve the above cubic equation in complete generality (short of citing Cardan's solution), we will simply note that every third order polynomial has one real root. Analytically, one sees that for $c \rightarrow -\infty$, the expression in parenthesis will eventually be negative and for $c \rightarrow \infty$, the expression will eventually be positive. Therefore, there must be some c such that the above expression vanishes.

After a bit of algebra, one sees that one can shift \mathcal{L} to the form,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4, \quad (3.2)$$

where,

$$\begin{aligned} \lambda &= \beta; \\ g &= (\alpha - \beta c); \\ m^2 &= \left(b - \alpha c + \frac{\beta c^2}{2} \right); \\ c &= \frac{-4\alpha \pm 2\sqrt{4\alpha^2 - 9\beta b}}{3\beta}. \end{aligned}$$